Invasion percolation with temperature and the nature of self-organized criticality in real systems

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In this paper we present a theoretical approach that allows us to describe the transition between critical and noncritical behavior when stocastic noise is introduced in extremal models with disorder. Namely, we show that the introduction of thermal noise in invasion percolation (IP) brings the system outside the critical point. This result suggests a possible definition of self-organized criticality systems as ordinary critical systems where the critical point corresponds to set to 0 one of the parameters. We recover both the IP and Eden models for $T\to 0$ and $T\to\infty$, respectively. For small T we find a dynamical second-order transition with correlation length diverging when $T\to 0$.

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The spontaneous development of complex and fractal structures has been studied on the basis of several models manifesting self-organized criticality (SOC) [1]. This concept is very intriguing and its very meaning has been highly debated. For instance, it has been noticed that the combination of different properties as, for example, stochastic noise and quenched disorder, usually destroys criticality. A big difficulty in order to clarify the real nature of SOC is the distance between the enormous amount of numerical results and the poor developement of systematic theoretical tools able to derive the average critical properties of the models directly from their formulation. Interesting theoretical approaches to SOC can be found in [2] and in [3].

In this paper we present a probabilistic approach to study systematically the effect of a stochastic noise on the spatio-temporal correlations of SOC models with quenched disorder. This method is called *generalized run time statistics* (GRTS) and generalizes for a *stocastic* dynamics the run time statistics (RTS) approach. Namely, the latter one has been originally formulated for the case of *deterministic* extremal dynamics in quenched disorder [4] as invasion percolation (IP) [5]. This generalization is very important because it allows us to study correlations and memory effects induced by the quenched disorder of the medium in any quasistatic stochastic growth process. In order to make clear the importance of this approach, we consider here one of the classical models of self-organization, the IP model. IP describes the displacement of a fluid in a disordered net of random throats due to another immiscible fluid pushed with a vanishing pressure rate. In this paper we study this model when a temperaturelike noise T is present. This generalization is important since through GRTS we can describe a more realistic case, where stochastic fluctuations affect the dynamics of invasion. Moreover, we can study analytically the robustness of SOC with respect to external solicitations [6,7]. The main result is that for any stochastic (thermal) noise a finite correlation length appears and the criticality is destroyed. This suggests a possible definition for SOC phenomena in real systems: *a system or a dynamical process is SOC if the critical value of the driving parameter is* 0, *instead of another real number*. This idea supports a similar view developed in [8]. The reason why such a value makes such a large difference is because the driving parameter of these process is always a ratio (grain of sand added with respect to the total number of sites for the sandpiles, sites whose ''value'' is changed for IP, DLA [9], Bak and Sneppen (BS) [10], etc.) and any value smaller than a certain threshold can be considered equal to 0. For this reason zero occupies a much larger region of the phase space than any other real number.

Let us recall the definition of the IP model [5]. In a lattice of size L a random number x_i, extracted from the uniform density $p_0(x)=1$ for $x\in[0,1]$, is assigned to each bond i. Let us call C_t the cluster of invaded bonds at time t (a finite connected set C_0 is fixed arbitrarily), and ∂C_t the interface of the cluster C_t. ∂C_t is the set of bonds not invaded but in contact with C_t. At time t the bond $i\in\partial C_t$ with the lowest x_i is invaded and added to the cluster C_t: $C_{t+1}=C_t\cup\{i\}$. The interface is consequently updated. The dynamics is repeated and stops when the cluster percolates the lattice.

This simple growth model develops spontaneously geometrical and dynamical critical features: (1) The asymptotic cluster is a fractal (i.e., it has an infinite correlation length) with fractal dimension $D_f\simeq 1.89$ in a $2d$ lattice, which is the same fractal dimension of the infinite cluster of percolation at the critical point. (2) The normalized histogram $\phi_t(x)$, of the interface variables, has the following asymptotic shape:

$$\phi_t(x)=\frac{1}{1-p_c}\theta(x-p_c), \tag{1}$$

while the initial shape is obviously $\phi_0(x)=1$. p_c coincides with the classical percolation threshold. (3) The asymptotic dynamics evolves for *critical avalanches*. Any bond i growing at time t is the *initiator* of its own avalanche. An *avalanche* is the temporal consecutive sequence of growth events geometrically and causally connected to the growth of the initiator (for a detailed definition of avalanche see, e.g., [11]). Note that, in the large time limit, the x of the initiator, due to the shape of the asymptotic histogram must be $x\leqslant p_c$. The size distribution $D(s;x)$ (where x is the random number of the initiator) of the avalanche has the following behavior:

$$D(s;x)=s^{-\tau}f(s^{\sigma}|x-p_c|), \tag{2}$$

where $f(x)=c>0$ for $x\ll 1$ and decays exponentially for $x\gg 1$ (i.e., for $s>s_0=|x-p_c|^{-1/\sigma}$). $\tau=1.57\pm 0.03$ and $\sigma=1-\tau+2/D_f=0.49\pm 0.03$. Note that if $x=p_c$, the size distribution is a power law as the characteristic size s_0 diverges [11]. As a consequence, these kind of avalanches are called *critical* avalanches.

We now generalize the IP model by introducing the presence of thermal noise. Numerical studies of an analogous application to the Bak-Sneppen model [10] can be found in [6], whilst the case of sandpiles has been considered in [7]. The first effect of a finite temperature T is that the deterministic dynamics becomes stochastic, such that the larger is the temperature the larger is the stochasticity. The definitions of C_t and ∂C_t in this model are the same as IP, but the growth rule is different: each bond $i\in\partial C_t$ has the following growth probability, depending on the realization of the quenched disorder:

$$\eta_{i,t}(\{x\}_{\partial C_t})=\frac{e^{-\beta x_i}}{\sum_{j\in\partial C_t} e^{-\beta x_j}}, \tag{3}$$

where $\beta=1/T$ and $\{x\}_{\partial C_t}$ is the realization of the quenched disorder on the interface ∂C_t. The larger is T the more $\eta_{i,t}$ is independent on x_i. Hereafter, we shall indicate with $\|C_t\|$ the number of bonds belonging to C_t, and with $\|\partial C_t\|$ the number of bonds belonging to ∂C_t.

It is important to study the two different limits $T\to\infty$ and $T\to 0$. In the first limit we have

$$\lim_{T\to\infty}\eta_{i,t}=\frac{1}{\|\partial C_t\|}, \tag{4}$$

where $\|\partial C_t\|$ is the total number of bonds belonging to the growth interface at time t. Equation (4) means that all the bonds on the interface have the same probability to grow. This model is well known and usually called the Eden model [12]: this dynamical growth generates a compact cluster (fractal dimension equal to the space dimension) with a rough surface. In the second limit we have

$$\lim_{T\to 0}\eta_{i,t}=\prod_{j\in\partial C_t-\{i\}}\theta(x_j-x_i), \tag{5}$$

where $\partial C_t-\{i\}$ means the interface ∂C_t minus the bond i. Equation (5) provides nothing else but the deterministic extremal growth rule of IP: $\eta_{i,t}=1$ if x_i is the extremal (minimum) value and zero otherwise. In this paper we address mainly the study of the behavior for small values of T, i.e., the transition of the model towards IP. In particular we will study the case of a $2d$ square bond lattice. We started by studying some Monte Carlo simulations of this model. The presence of the temperature introduces a characteristic length $\xi(T)$, the effect of which is quite clear in Fig. 1 where percolating clusters for different values of $\beta=1/T$ are shown. The differences between the clusters can be explained by characterizing qualitatively the dynamics of growth.

For any value of T, a characteristic time $t^*(T)$ exists such that, for $t<t^*(T)$ the dynamics of the model is the IP dynamics, i.e., even if the dynamical rule given by Eq. (3) is not deterministic, the effect of stochasticity is still negligible and the effective dynamics is almost extremal. On the other hand, for $t>t^*(T)$ the effect of the stochastic noise begins to be more and more important and the deviation from IP and then from fractality, becomes larger. If we suppose that $T\ll 1$, and then $t^*(T)\gg 1$, it is clear that $t^*(T)$ represents the correlation time of the system. Since one bond is removed for each time step, $t^*(T)$ represents also the number of bonds $s_0(T)$ in a correlated region of the cluster when $t\gg t^*(T)$. This is in agreement with the idea that at $T>0$ IP is the repulsive fixed point of the dynamics under a spatio-temporal coarse-graining transformation, and the Eden model is the trivial attractive fixed point characterized by $T\to\infty$. These features can be checked by looking at the dynamical evolution of the histogram $\phi_t(x)$. Obviously $\phi_0(x)=1$; for $t<t^*(T)$ as previously noted, the evolution is the same as the IP, that is $\phi_t(x)$ evolves in the step function given by Eq. (1). At $t=t^*(T)$, $\phi_t(x)$ is a smoothened step function (the size of the smoothened interval around p_c increases with T). For $t>t^*(T)$, because of stochasticity, the growth of bonds with x much larger than p_c are permitted and the histogram $\phi_t(x)$ shifts towards high values of x. We have measured through simulations $t^*(T)$ by measuring the time step when $\phi_t(x)$ start to shift and we obtain the scaling law $t^*(T)\equiv s_0(T)\sim T^{-\gamma}$ with $\gamma=1.9\pm 0.2$. In the following we find the same behavior theoretically and we link it to the correlation length of the structure.

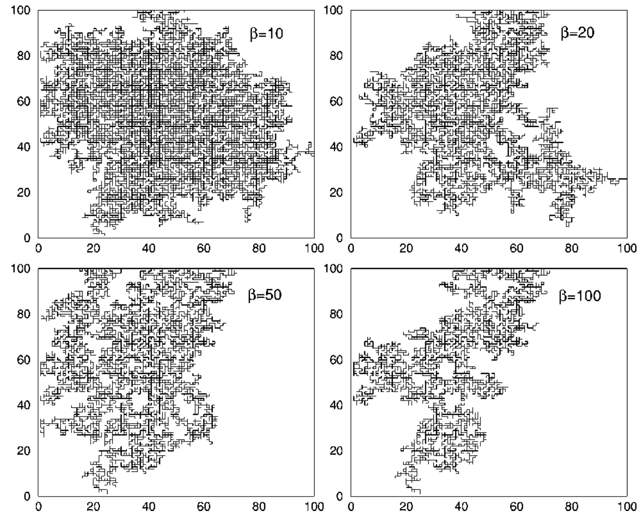

FIG. 1. Different percolating clusters for different values of $\beta=1/T$ in a lattice of linear size $L=100$. The ''fractality'' increases with β.

To study the model, we formulate the generalization to stochastic growth dynamics of the run time statistics (RTS) [4,13] that we call generalized run time statistics (GRTS). The usual RTS is a probabilistic technique based on the concept of conditional probability, introduced to study IP-like dynamics, i.e., deterministic extremal dynamics with quenched disorder. With the GRTS approach we can solve the following problem: suppose we fix the time-ordered path C_t followed by the dynamics, and we ignore the realization of the disorder: then we can compute the joint probability density function $P_t(\{x\}_{\partial C_t})$ of all the variables x_i of the bonds i belonging to the interface ∂C_t, conditioned to the history C_t. Furthermore, we can compute the conditioned probability of the next growth steps. This joint probability density function (PDF) $P_t(\{x\}_{\partial C_t})$ plays a central role, since from it we can compute the probability (conditioned to the whole past history, i.e., to all the previous steps of the path) of any possible next growth step. After that, we update consequently the joint probability density itself obtaining $P_{t+1}(\{x\}_{\partial C_{t+1}})$. Here we expose an approximated version of GRTS. The approximation consists of assuming that at any time step the PDF can be written as the product of single-bond density functions $p_{k,t}(x_k)$

$$P_t(\{x\}_{\partial C_t})=\prod_{k\in\partial C_t} p_{k,t}(x_k).$$

This means that one is assuming that all the information about the history can be contained in the set of effective single-bond density functions. Usually this is not the case; one can show that the information about the dynamical history generates correlations among the interface variables [14]. However, it can be seen [15] that this approximation works very well even for IP where the the effect of this correlation, because of the extremal nature of the dynamics, is the maximum.

Starting from the PDF's we want to compute the conditional probability $\mu_{i,t}$ that a certain bond $i\in\partial C_t$ grows at time t. Let us suppose we know the ''effective'' one bond PDF $p_{k,t}(x_k)$ for each $k\in\partial C_t$. The functions $p_{k,t}(x_k)$ are determined by the whole past history up to time t [obviously, for $t=0$ each $p_{k,t}(x)=p_0(x)=1$ as there is no information yet on the dynamics]. Knowing the functions $p_{k,t}(x_k)$, the conditioned probability $\mu_{i,t}$ are given by

$$\mu_{i,t}=\int_0^1\ldots\int_0^1\prod_{k\partial C_t}[dx_k p_{k,t}(x_k)]\frac{e^{-\beta x_i}}{\sum_{k\in\partial C_t}e^{-\beta x_k}}. \tag{6}$$

The set of μ's, for each $i\in\partial C_t$, provides the growth probability distribution (GPD) conditioned to the past dynamical history up to time t. For each of these growth events, we may update the old effective PDF's $p_{k,t}(x)$, ''conditioning'' them to the knowledge of the last step at time t. In order to do that, we have to distinguish three cases: (a) the last grown bond i, (b) the other bonds j belonging to ∂C_t, and finally (c) the bonds just entered in the new interface ∂C_{t+1} because of the growth of i.

(a) In this case, i does not belong to ∂C_{t+1}. For this reason, we use the new symbol $m_{i,t+1}(x)$ for its PDF at time $t+1$,

$$m_{i,t+1}(x)=\frac{1}{\mu_{i,t}}\int_0^1\ldots\int_0^1\prod_{k\partial C_t}[dx_k p_{k,t}(x_k)]\times\frac{e^{-\beta x_i}}{\sum_{k\in\partial C_t}e^{-\beta x_k}}\delta(x_i-x). \tag{7}$$

(b) In this case we have

$$p_{j,t+i}(x)=\frac{1}{\mu_{i,t}}\int_0^1\ldots\int_0^1\prod_{k\partial C_t}[dx_k p_{k,t}(x_k)]\times\frac{e^{-\beta x_i}}{\sum_{k\in\partial C_t}e^{-\beta x_k}}\delta(x_j-x). \tag{8}$$

(c) Finally, $p_{j,t+i}(x)=p_0(x)=1$; let us call $n_{i,t}$ the number of these bonds. Note that the following relations hold: $\|C_t\|=t$ and $\|\partial C_{t+1}\|=\|\partial C_t\|+n_{i,t}-1$. Hereafter we call Ω_t and n_t the average values, over different dynamical realizations, respectively, of $\|C_t\|$ and $n_{i,t}$.

Using Eqs. (6)–(8) and the rule that bonds just entered the interface have simply $p_0(x)=1$ as an ''effective'' density function, we can describe from a conditional probability point of view any possible dynamical history, knowing only $p_0(x)$ and the dynamical rule given by Eq. (3). In [13,16] the $T=0$ case of GRTS was formulated and used to study IP, evaluating both D_f and τ. Now we use this generalized approach to study the transition towards IP (*stochastic-extremal* transition). First of all the histogram $h_t(x)$ is introduced. $h_t(x)$ is the distribution of x's on the interface at time t. That is, $h_t(x)dx$ provides the number of interface bonds at time t with x belonging to th intervall $[x,x+dx]$.

If we fix a dynamical history up to time t, we can write

$$h_t(x)=\sum_{i\in\partial C_t}p_{i,t}(x),$$

where the functions $p_{i,t}(x)$ must be evaluated through the algorithm provided by Eqs. (6)–(8) for the fixed history. Note that $\int_0^1 dx h_t(x)=\|\partial C_t\|$. Since the disorder is quenched, the dynamical equation for $h_t(x)$ is

$$h_{t+1}(x)=h_t(x)-m_{i,t+1}(x)+n_{i,t}p_0(x). \tag{9}$$

It is convenient to study the normalized histogram $\phi_t(x)$, defined as $\phi_t(x)=h_t(x)/\|\partial C_t\|$. Since (as for IP) $\phi_t(x)$ is an almost self-averaging quantity for small T, we can take the average of Eq. (9) over all the possible histories in order to evaluate $\phi_t(x)$. After some algebra and approximations, one can write the following equations:

$$\Omega_{t+1}\phi_{t+1}(x)=\Omega_t\phi_t(x)-\Omega_t\phi_t(x)\frac{1}{1+\Omega_t e^{\beta(x-1/n_t)}}+n_t, \tag{10}$$

where $\Omega_{t+1}=\Omega_t+n_t-1$. To obtain Eq. (10), we have assumed that $e^\beta\gg\Omega_t\gg 1$. Clearly the dynamical evolution of the histogram is strictly related to that of n_t; in IP for $t\gg 1$ we have $n_t\simeq 1/p_c$ [4]. Because of the quasistaticity of the dynamics, the evolution of $\phi_t(x)$ is very slow [i.e., $|\phi_{t+1}(x)-\phi_t(x)|/\phi_t(x)\ll 1$]. Consequently, from Eq. (10) for $t\gg 1$, we can write approximatively

$$\phi_t(x)\simeq\frac{n_t}{n_t-1+\frac{1}{\frac{1}{\Omega_t}+e^{\beta(x-1/n_t)}}}. \tag{11}$$

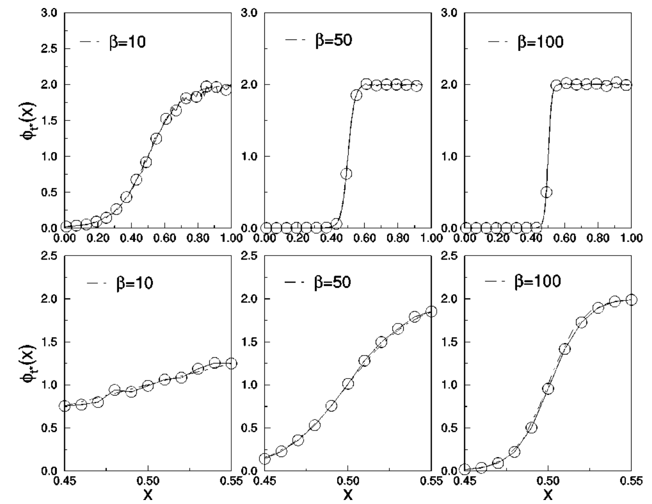

FIG. 2. Different histograms at $t=t^*(T)$ for three different values of $\beta=1/T$ $(=10,50,100)$. The three top figures provide the histograms in the whole interval $x\in[0,1]$; the three bottom ones provide the same quantities in the reduced window $x\in[0.45,0.55]$. The larger β is the more IP-like the histogram is. The dashed line represents Eq. (12); numerical data are represented by empty circles connected by a filled line.

$\phi_t(x)$ is a smoothened step function around $x=1/n_t$ with $\Delta x\sim T$. For $t=t^*(T)\gg 1$ we use the IP relation $n_t\simeq 1/p_c$, since dynamics is IP-like [13,17]. Then

$$\phi_{t*}(x)\simeq \frac{1}{1-p_c+\dfrac{p_c}{\dfrac{1}{\Omega_{t*}}+e^{\beta(x-p_c)}}}. \qquad (12)$$

This function differs from Eq. (1) only in an interval of extension $\Delta x\sim T$ just around $x=p_c$. The agreement between this function and the numerical data is very good for a wide range of T (Fig. 2). Note that the relation $n_t\simeq 1/p_c$ is true at any time $(t\gg 1)$ only in the case $T=0$. For $T>0$, because of stochastic noise, $n_t\to 1$ for $t\gg t^*(T)$ and the cluster becomes compact (it can be shown that n_t-1 represents the asymptotic value of the ratio between the interface number $\|\partial C_t\|$ and the cluster number $\|C_t\|$).

From Eq. (12) and from the exponents of IP, we can obtain the behaviors of $s_0(T)$ and $\xi(T)$ at small T. In IP an avalanche, with an initiator with $x=p_c-\Delta x$, has a typical size $s_0(\Delta x)\sim\Delta x^{-1/\sigma}$. Here we have a natural value $\Delta x\sim T$ even for the maximal sequence of correlated growth events. Hence $s_0(T)\sim T^{-1/\sigma}=T^{-\gamma}$ with $\gamma=2.0\pm0.1$ in agreement with the simulations. For the fractality of IP, we have $s_0(T)\sim\xi(T)^D$, hence $\xi(T)\sim T^{-\nu}$ with $\nu=\gamma/D=1.10\pm0.05$.

In conclusion, we presented here a general probabilistic approach, the GRTS, for a quasistatic stochastic dynamical model in a medium with quenched disorder. Through this method we study memory effects and temporal correlations induced by the disorder. In particular, we describe the IP model where a temperaturelike parameter T is introduced. Through the GRTS it is found that the larger the stochastic noise, the lower the memory effects and the weaker the geometrical correlations developed during the dynamics. Namely, the model produces structures that are fractal and self-organized only by tuning this parameter to 0, otherwise a finite correlation length exists. This behavior (similar to that observed for the BS model Ref. 11 in [6]) supports the hypothesis that SOC models are closely related to ordinary critical systems, where parameters have to be tuned to their critical value.

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